ANSWER KEY — TIDES & ENERGIES OF ORBITS

Rotational Energy & Tidal Friction | For teacher use only

PAGE 1 — EARTH'S SPIN ENERGY (WE DO)

Earth's Rotational KE

Given: $M = 5.97 \times 10^{24}$ kg, $R = 6.37 \times 10^{6}$ m, $T = 24.0$ hr $= 86,400$ s

(a) Find ω:

$\omega = \frac{2\pi}{T} = \frac{2\pi}{86{,}400} = 7.27 \times 10^{-5}$ rad/s

(b) Find I_{Earth}:

$I = \frac{2}{5}MR^2 = 0.4 \times (5.97 \times 10^{24}) \times (6.37 \times 10^{6})^2$

$= 0.4 \times (5.97 \times 10^{24}) \times (4.06 \times 10^{13})$

$= 9.70 \times 10^{37}$ kg·m^2

(c) Find K_{rot}:

$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(9.70 \times 10^{37})(7.27 \times 10^{-5})^2$

$= \frac{1}{2}(9.70 \times 10^{37})(5.29 \times 10^{-9})$

$= 2.57 \times 10^{29}$ J

Scale: Human civilization uses ~6×10^{20} J/year, so Earth's spin contains ~430 million years of global energy consumption!

PAGE 2 — RATE OF KE LOSS (WE DO)

Tidal Braking Power

(a) Find $\Delta\omega$ per century from 2.3 μs day lengthening:

$\Delta T = 2.3 \times 10^{-5}$ s per century, $\quad T = 86,400$ s

From $\omega = 2\pi/T$, small changes: $\Delta\omega = -\omega\frac{\Delta T}{T}$

$\Delta\omega = -(7.27 \times 10^{-5}) \times \frac{2.3 \times 10^{-5}}{86{,}400}$

$\Delta\omega = -(7.27 \times 10^{-5}) \times (2.66 \times 10^{-10})$

$\Delta\omega = -1.93 \times 10^{-14}$ rad/s per century

(b) Energy lost per century and power:

$\Delta K = I\omega\Delta\omega$

$\Delta K = (9.70 \times 10^{37})(7.27 \times 10^{-5})(-1.93 \times 10^{-14})$

$\Delta K = -1.36 \times 10^{20}$ J per century (energy lost)

Power: $P = \frac{|\Delta K|}{\Delta t} = \frac{1.36 \times 10^{20}}{100 \text{ years} \times 3.156 \times 10^{7} \text{ s/year}}$

$P = 4.3 \times 10^{12}$ W $= 4.3$ TW

This matches the real estimate of ~3.7 TW! (Order of magnitude agreement is excellent.)

Energy Accounting (You Do)

(a) Percentage breakdown of 3.7 TW total dissipation:

Ocean heating: 3.2 TW → $\frac{3.2}{3.7} =$ **86.5%**

Moon's orbit: 0.5 TW → $\frac{0.5}{3.7} =$ **13.5%**

(b) Total energy transferred in one year:

$E = P \times t = (3.7 \times 10^{12} \text{ W}) \times (3.15 \times 10^{7} \text{ s})$

$= 1.17 \times 10^{20}$ J per year

(c) Why tidal energy is NOT renewable (from energy conservation perspective):

Tidal energy comes from Earth's rotational kinetic energy and the Moon's orbital energy. As we extract tidal energy (by building dams/power plants), we drain these reservoirs faster, slowing Earth's rotation and the Moon's orbit even more. Energy is NOT created by tides — it's transferred from spin/orbital motion to thermal energy. Once Earth's rotation slows to match the Moon's orbital period (in ~50 billion years), there will be no more tidal friction and tides will stop. Tidal energy is therefore "mining" ancient rotational energy, not a self-renewing source.

PAGE 3 — ORBITAL ENERGY PARADOX

Speed-Energy Puzzle (We Do)

Using $v = \sqrt{GM/r}$ **with** $GM = 3.99 \times 10^{14}\ \mathrm{m^3/s^2}$**:**

Orbit	v (m/s)	K/m (J/kg)
LEO (400 km)	**7,680**	$\frac{1}{2}(7680)^2 = 2.95 \times 10^7$
Geostationary	**3,075**	$\frac{1}{2}(3075)^2 = 4.73 \times 10^6$

(a) Which orbit has MORE kinetic energy per kilogram?

LEO has 6.2 times more KE per kg because it's moving much faster (lower orbit).

(b) Yet to reach geostationary from LEO, you must ADD energy. Why?

Orbital energy = $KE + PE$**. In a higher orbit, you have much MORE gravitational potential energy (PE becomes less negative). Even though KE decreases, the gain in PE is larger, so total orbital energy increases. The tether/thruster must do work against gravity to lift the orbit. Most of that work goes into increasing PE, not KE. This is the key misconception — "adding energy to an orbit" doesn't mean speeding up; it means moving to a less bound (higher PE) orbit where you move slower but have higher total energy.**

Angular Momentum + Energy Connection (You Do)

(a) If Moon's orbital radius increases, does L_{Moon} **increase or decrease?**

$L = m\sqrt{GMr}$ — as r increases, $\sqrt{r}$ increases, so **L increases**.

(b) If orbital radius increases, does orbital KE increase or decrease?

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(GM/r)$ — as r increases, KE **decreases**.

(c) One-sentence summary:

As tidal friction transfers angular momentum from Earth to the Moon, the Moon moves outward, gets slower, gains angular momentum and orbital energy, and loses kinetic energy.

PAGE 4 — PRACTICE & EXIT TICKET

1. Neutron Star Rotational KE

Given: Mass $= 2.0 \times 10^{30}\ \mathrm{kg}$, radius = 10 km = $10^4\ \mathrm{m}$, spin = 30 rev/s

(a) Find I **and** ω**:**

$I = \frac{2}{5}MR^2 = 0.4 \times (2.0 \times 10^{30})(10^4)^2 = 0.4 \times 2.0 \times 10^{38} = 8.0 \times 10^{37}\ \mathrm{kg{\cdot}m^2}$

$\omega = 30\ \mathrm{rev/s} \times 2\pi = 188\ \mathrm{rad/s}$

(b) Find K_{rot}**:**

$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8.0 \times 10^{37})(188)^2 = 0.5 \times 8.0 \times 10^{37} \times 3.53 \times 10^4$

$= 1.41 \times 10^{42}\ \mathrm{J}$

Compare to Earth: $\frac{1.41\times10^{42}}{2.57\times10^{29}} = 5.5\times10^{12}$ **times larger!** A neutron star spinning 30 times per second has more rotational energy than Earth by a factor of trillions.

(c) What acts as "tidal friction" for neutron stars?

Magnetic braking. Neutron stars emit beams of radiation from their magnetic poles (pulsars). As they spin, they lose energy via electromagnetic radiation. This is how pulsars slow down over time.

2. Classify Energy Transfers

Scenario	Loses K_{rot}	Energy goes to...
Brakes slow a spinning flywheel	**Flywheel**	**Heat (friction in brakes)**
Earth's tides slow Earth's rotation	**Earth**	**Heat + Moon's orbital energy (97% heat, 3% Moon)**
Jupiter's tides flex Io's interior	**Io's orbital motion + spin**	**Heat inside Io (drives volcanism)**
Friction brings two stacked disks to same ω	**Both disks**	**Heat (friction between disks)**

3. KE Lost in Rotational "Collision"

Given: Disk A: $I_A = 0.40\ \text{kg·m}^2, \omega_A = 10\ \text{rad/s}$. Disk B: $I_B = 0.60\ \text{kg·m}^2, \omega_B = 0$.

(a) Find ω_f:

$L_i = I_A\omega_A + I_B\omega_B = (0.40)(10) + 0 = 4.0\ \text{kg·m}^2/\text{s}$

$L_f = (I_A + I_B)\omega_f = (1.0)\omega_f = 4.0$

$\omega_f =$ **4.0 rad/s**

(b) Calculate KE before and after:

$K_i = \frac{1}{2}I_A\omega_A^2 = \frac{1}{2}(0.40)(10)^2 =$ **20 J**

$K_f = \frac{1}{2}(1.0)(4.0)^2 =$ **8.0 J**

$\Delta K = K_f - K_i =$ **-12 J lost**

(c) Where did lost KE go? Why is this like an inelastic collision?

> **The lost 12 J became heat/sound/deformation at the contact between disks where friction brought them to the same ω. This IS exactly like an inelastic collision: momentum (or angular momentum) is conserved, but kinetic energy is NOT — it's dissipated as heat. In a collision two objects stick together; here, two rotating disks synchronize. Same physics.**

Exit Ticket: Is Tidal Energy Really Renewable?

No. Tidal energy comes from Earth's rotational KE and the Moon's orbital energy, which are finite reserves. As tides are harvested, we drain these reserves faster, further slowing Earth's rotation. The tides will eventually stop when Earth's rotation period matches the Moon's orbital period (~50 billion years from now). At that point, the "battery" is dead. Unlike solar or wind energy (powered by continuous external sources: the Sun), tidal energy is mining ancient momentum. It may be a very long-lasting resource, but it is definitively NOT renewable.