ANSWER KEY — ANGULAR MOMENTUM IN ORBIT

Orbits, Tides & Angular Momentum Conservation | For teacher use only

PAGE 1 — COMPARING ORBITAL ANGULAR MOMENTUM (WE DO)

ISS vs. Geostationary Satellite

(a) ISS orbital velocity and angular momentum:

Using $v = \sqrt{GM/r}$ with $GM = 3.99 \times 10^{14}\ \mathrm{m^3/s^2}$

$v_{\mathrm{ISS}} = \sqrt{\frac{3.99\times10^{14}}{6.78\times10^{6}}} = \sqrt{5.887 \times 10^{7}} =$ **7670 m/s**

$L_{\mathrm{ISS}} = mvr = (420,000)(7670)(6.78 \times 10^{6}) =$ **2.18 \times 10^{16} \text{ kg·m}^2\text{/s}\)**

(b) Geostationary satellite:

$v_{\mathrm{geo}} = \sqrt{\frac{3.99\times10^{14}}{4.22\times10^{7}}} = \sqrt{9.456 \times 10^{6}} =$ **3075 m/s**

$L_{\mathrm{geo}} = (3000)(3075)(4.22 \times 10^{7}) =$ **3.90 \times 10^{14} \text{ kg·m}^2\text{/s}\)**

(c) Per unit mass, which has more L/m?

$\frac{L_{\mathrm{ISS}}}{m_{\mathrm{ISS}}} = v_{\mathrm{ISS}} \cdot r_{\mathrm{ISS}} = (7670)(6.78 \times 10^{6}) = 5.20 \times 10^{10}\ \mathrm{m^2/s}$

$\frac{L_{\mathrm{geo}}}{m_{\mathrm{geo}}} = v_{\mathrm{geo}} \cdot r_{\mathrm{geo}} = (3075)(4.22 \times 10^{7}) = 1.30 \times 10^{11}\ \mathrm{m^2/s}$

Geostationary has more L/m (by factor 2.5). Because $L = m\sqrt{GMr}$, the factor of r dominates — even though the geosat is slower, it's much farther out.

PAGE 2 — EARTH'S SPIN & THE CLASSIFICATION TABLE

Earth's Spin Angular Momentum (We Do)

(a) Find ω in rad/s:

$T = 24.0\ \mathrm{hr} = 86,400\ \mathrm{s}$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{86{,}400} = 7.27 \times 10^{-5}\ \mathrm{rad/s}$

(b) Find I_{Earth}:

$I = \frac{2}{5}MR^2 = \frac{2}{5}(5.97 \times 10^{24})(6.37 \times 10^{6})^2$

$= 0.4 \times 5.97 \times 10^{24} \times 4.06 \times 10^{13}$

$= 9.70 \times 10^{37}\ \mathrm{kg{\cdot}m^2}$

(c) Find L_{spin}:

$L = I\omega = (9.70 \times 10^{37})(7.27 \times 10^{-5}) = 7.05 \times 10^{33}\ \mathrm{kg{\cdot}m^2/s}$

Earthquake Calculation (You Do)

(a) Show that $\omega_{new} = \frac{\omega}{1-\delta}$:

$L = I\omega = \mathrm{constant}$

$I\omega = I_{\mathrm{new}}\omega_{\mathrm{new}} = I(1-\delta)\omega_{\mathrm{new}}$

$\omega = (1-\delta)\omega_{\mathrm{new}}$

$\omega_{\mathrm{new}} = \frac{\omega}{1-\delta}$ ✓

(b) Find fractional change δ from the 2011 quake shortening day by 1.8 µs:

Day shortened: $\Delta T = 1.8 \times 10^{-6}$ s out of $T = 86,400$ s

For small δ, $\Delta\omega/\omega \approx \delta$

$\delta \approx \frac{\Delta T}{T} = \frac{1.8\times10^{-6}}{86{,}400} = 2.08 \times 10^{-11}$

Classification Table (Quick)

For each event: $L = I\omega$ is conserved when no external torque acts (internal Earth rearrangements).

Event	Effect on I	Effect on ω	Day gets...
Earthquake shifts mass toward axis	**Decreases**	**Increases**	**Shorter**
Glaciers melt, water flows to equator	**Increases**	**Decreases**	**Longer**
Asteroid hits Earth in direction of spin	**No change**	**Increases (external torque)**	**Shorter**
All humans walk to the equator and stand there	**Increases**	**Decreases**	**Longer**

PAGE 3 — TIDAL RECESSION ANALYSIS

Conservation Connection (Analysis)

(a) If the Moon's orbital L increases, what must happen to Earth's spin L?

Total $L = I_{\text{Earth}}\omega_{\text{Earth}} + L_{\text{Moon orbital}} = \text{constant}$

Earth's spin L must DECREASE (angular momentum is transferred from Earth's spin to Moon's orbit)

(b) So what happens to ω_{Earth}?

Earth's I doesn't change. $L = I\omega$. If L decreases, then ω_{Earth} **must DECREASE** — Earth's rotation slows down.

(c) Why was the Moon tidally locked first (explain in terms of $\tau = I\alpha$):

> **The Moon is much less massive and smaller than Earth, so it has much smaller moment of inertia I. The same tidal torque τ acts on both bodies (by Newton's 3rd law), so $\alpha = \tau/I$ is much larger for the Moon. Larger angular deceleration means the Moon slowed first. The Moon's orbital period now equals its spin period (tidally locked), but Earth is still spinning faster than the Moon orbits — so the process continues, transferring more angular momentum outward.**

PAGE 4 — EXIT TICKET & HOMEWORK

Exit Ticket: Asteroid Impact

(a) Calculate ΔL delivered to Earth:

$\Delta L = mvR = (5.0 \times 10^{12})(20 \times 10^3)(6.37 \times 10^6)$

$\Delta L = (5.0 \times 10^{12})(2.0 \times 10^4)(6.37 \times 10^6)$

$\Delta L = 6.37 \times 10^{23}\ \text{kg·m}^2/\text{s}$

(b) Fractional change in L:

$\frac{\Delta L}{L_{\text{Earth}}} = \frac{6.37\times10^{23}}{7.1\times10^{33}} = 8.96 \times 10^{-11}$ **or about** 9×10^{-11}

(c) Day gets shorter or longer? By how many seconds?

Asteroid adds L in direction of spin → ΔL is positive → ω increases → **day gets SHORTER**

Fractional change in ω: $\frac{\Delta\omega}{\omega} = \frac{\Delta L}{L} = 9 \times 10^{-11}$

$\Delta T/T = 9 \times 10^{-11}$, so $\Delta T = 86,400 \times 9 \times 10^{-11} = 7.8 \times 10^{-6}\ \text{s}$ **or ~8 nanoseconds** (imperceptible!)

Homework

1. Moon vs. Earth

(a) Calculate L_{Moon}:

$L = mvr = (7.35 \times 10^{22})(1022)(3.84 \times 10^8)$

$= 2.88 \times 10^{34}\ \text{kg·m}^2/\text{s}$

(b) Compare to Earth's spin $L = 7.1 \times 10^{33}$:

$\frac{L_{\text{Moon}}}{L_{\text{Earth}}} = \frac{2.88 \times 10^{34}}{7.1 \times 10^{33}} =$ **4.05 or about 4 times larger**

The Moon's orbital angular momentum is ~4× larger than Earth's spin angular momentum!

2. The 3.8 cm Recession

(a) Percentage increase in orbital radius per year:

$\frac{\Delta r}{r} = \frac{0.038}{3.84 \times 10^8} = 9.9 \times 10^{-11}$ **or about** 10^{-10}

(b) Years until Moon is 10% farther:

Need: $\Delta r / r = 0.10$

$t = \frac{0.10}{9.9 \times 10^{-11} \text{ per year}} = 1.01 \times 10^{10}$ years **or ~10 billion years**

3. Satellite Boost (Conceptual)

A satellite moving to a higher orbit speeds down because $v = \sqrt{GM/r}$ — larger r means smaller v. However, $L = mvr = m\sqrt{GMr}$ — the r factor grows faster than v shrinks, so L increases. **The satellite loses kinetic energy but gains gravitational potential energy. The potential energy gain exceeds the KE loss, so total orbital energy increases. It's counterintuitive because KE and L move in opposite directions here.**

4. Will the Moon Escape?

No, the Moon will NOT escape. Tidal locking eventually brings the Moon's orbital period equal to Earth's rotation period. At that point, there's no more relative motion, no more tidal friction, and the process stops. The Moon stays in a stable but higher orbit. (In reality, the Sun's tidal effects on the Earth-Moon system are also important for the very long-term future.)