ANSWER KEY — ROTATION DAY 7: ROTATIONAL INERTIA

Moment of Inertia | For teacher use only

WARM-UP (PAGE 1)

Prediction: (B) Disk. The disk reaches the bottom first.

Reasoning: Both objects convert gravitational PE to translational and rotational KE. Since $I_{\text{hoop}} > I_{\text{disk}}$, the hoop "wastes" more energy on spinning, leaving less for translational motion. The disk wins the race.

WE DO: POINT MASSES ON A ROD (PAGE 1)

(a) Calculate I for the system

$$I = \sum m_i r_i^2$$

$$I = 2.0(0.10)^2 + 1.0(0.20)^2 + 3.0(0.30)^2$$

$$I = 0.02 + 0.04 + 0.27 = \boxed{0.33\ \text{kg·m}^2}$$

I = 0.33 kg·m²

(b) If the 3.0 kg mass moves to 0.10 m, what is I_new?

$$I_{\text{new}} = 2.0(0.10)^2 + 1.0(0.10)^2 + 3.0(0.10)^2$$

$$I_{\text{new}} = 0.02 + 0.01 + 0.03 = \boxed{0.06\ \text{kg·m}^2}$$

I_new = 0.06 kg·m²

(c) Which arrangement is easier to spin up? Why?

The second arrangement (I_new = 0.06 kg·m²) is easier to spin up.

Why: The rotational inertia is much smaller. Since $\alpha = \tau/I$, smaller I means larger angular acceleration for the same torque. Moving the 3.0 kg mass closer to the axis dramatically reduces I because of the r^2 dependence.

RANKING TASK (PAGE 2)

Ranking (largest to smallest I):

(D) > (B) > (A) > (C)

Point mass at distance R > Hoop > Disk > Solid sphere

All have the same mass and characteristic dimension. The formula $I \propto cMR^2$ shows:

- Point mass: $c = 1$
- Hoop: $c = 1$
- Disk: $c = 0.5$
- Sphere: $c = 0.4$

This confirms the ranking. Both point mass and hoop have $c = 1$, but the point mass dominates the ranking presented.

HOOP VS. DISK (PAGE 2)

(a) Which has larger rotational inertia?

$$I_{\text{hoop}} = MR^2 = MR^2$$

$$I_{\text{disk}} = \frac{1}{2}MR^2$$

The hoop has larger I.

(b) Which resists angular acceleration more?

The hoop resists angular acceleration more. Since $\alpha = \tau/I$, larger I means smaller α for the same τ.

(c) Which object has more translational speed at the bottom?

Energy conservation: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
With $v = \omega R$: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{I}{R^2}\right)v^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$
For hoop: $v = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$
For disk: $v = \sqrt{\frac{2gh}{1+0.5}} = \sqrt{\frac{4gh}{3}}$
The disk has higher translational speed at the bottom.

(d) Does the race result depend on mass? On radius?

No, the ranking does not depend on mass or radius.

The formula $v = \sqrt{\frac{2gh}{1+c}}$ depends only on the shape constant $c = I/(MR^2)$, not on M or R. An enormous disk and a tiny disk will always tie in a race on the same ramp.

MASS DISTRIBUTION RANKING (PAGE 3)

Calculate I for each system:

(A): $I_A = 4.0(0.50)^2 = 1.00 \text{ kg·m}^2$
(B): $I_B = 2.0(0.30)^2 + 2.0(0.70)^2 = 0.18 + 0.98 = 1.16 \text{ kg·m}^2$
(C): $I_C = 4.0(0.20)^2 = 0.16 \text{ kg·m}^2$
(D): $I_D = 1.0(0.20)^2 + 1.0(0.40)^2 + 1.0(0.60)^2 + 1.0(0.80)^2$
$= 0.04 + 0.16 + 0.36 + 0.64 = 1.20 \text{ kg·m}^2$

Ranking (largest to smallest): D > B > A > C

Justification: System D spreads its mass over the longest distances (especially the 0.80 m masses), so it has the largest I. System B keeps most mass closer to the axis except for the two 0.70 m masses. System A puts all mass at one distance. System C concentrates all mass very close to the axis.

PROPORTIONAL REASONING (PAGE 3)

(a) If r doubles (mass stays the same), by what factor does I change?

Since $I = mr^2$, when $r \to 2r$:

$$I_{\text{new}} = m(2r)^2 = 4mr^2 = 4I_{\text{old}}$$

I increases by a factor of 4.

(b) If m doubles (distance stays the same), by what factor does I change?

Since $I = mr^2$, when $m \to 2m$:

$$I_{\text{new}} = 2mr^2 = 2I_{\text{old}}$$

I increases by a factor of 2.

(c) Move from 0.30 m to 0.90 m. By what factor does I change? Verify with calculations.

Distance ratio: $\frac{0.90}{0.30} = 3$, so $I_{\text{new}} = I_{\text{old}} \times 3^2 = 9I_{\text{old}}$
Verify:

$$I_1 = 2.0(0.30)^2 = 2.0(0.09) = 0.18 \text{ kg·m}^2$$

$$I_2 = 2.0(0.90)^2 = 2.0(0.81) = 1.62 \text{ kg·m}^2$$

$$\frac{I_2}{I_1} = \frac{1.62}{0.18} = 9$$

I increases by a factor of 9.

HOMEWORK (PAGE 4)

1. Point Masses in a Square

Square with side length 0.40 m. Distance from center to corner:

$$r = \frac{\text{diagonal}}{2} = \frac{0.40\sqrt{2}}{2} = 0.283 \text{ m}$$

All four masses are at the same distance from the center:

$$I = 4 \times 0.50 \times (0.283)^2 = 4 \times 0.50 \times 0.0800 = 0.16 \text{ kg·m}^2$$

I = 0.16 kg·m²

2. Shape Comparison: Hoop vs. Disk

(a) Calculate I for each

Hoop: $I_{\text{hoop}} = MR^2 = 3.0(0.20)^2 = 0.12 \text{ kg·m}^2$
Disk: $I_{\text{disk}} = \frac{1}{2}MR^2 = \frac{1}{2}(3.0)(0.20)^2 = 0.06 \text{ kg·m}^2$

I_hoop = 0.12 kg·m²; I_disk = 0.06 kg·m²

(b) Find α for each with τ = 0.60 N·m

Using $\tau = I\alpha$, so $\alpha = \tau / I$:
Hoop: $\alpha_{\text{hoop}} = \frac{0.60}{0.12} = 5.0 \text{ rad/s}^2$
Disk: $\alpha_{\text{disk}} = \frac{0.60}{0.06} = 10 \text{ rad/s}^2$

α_hoop = 5.0 rad/s²; α_disk = 10 rad/s²

3. Axis Location Matters

(a) Calculate I for each axis

Through center: $I_{\text{center}} = \frac{1}{12}ML^2 = \frac{1}{12}(2.0)(1.0)^2 = 0.167\ \text{kg·m}^2$
Through end: $I_{\text{end}} = \frac{1}{3}ML^2 = \frac{1}{3}(2.0)(1.0)^2 = 0.667\ \text{kg·m}^2$

I_center = 0.167 kg·m²; I_end = 0.667 kg·m²

(b) Why is I_end four times larger?

When the axis is at the center, half the mass is on each side, with average distance $L/4$. When the axis is at the end, all the mass is distributed outward, with average distance $L/2$. The axis at the end puts more mass far away, so I is much larger. The exact factor of 4 comes from the formulas: $I_{\text{end}}/I_{\text{center}} = (\frac{1}{3})/(\frac{1}{12}) = 4$.

4. Conceptual: Baseball Bat

Holding the bat by the barrel (choked up) makes it harder to swing.

When you hold it normally (by the handle), the heavy barrel is far from your wrist, creating a large rotational inertia. The bat is hard to accelerate but once moving, it's harder to stop. When you choke up (hold near the barrel), the center of mass is much closer to your wrist, reducing I. The bat becomes easier to accelerate and control. (This is why batters "choke up" in tight situations.)