ANSWER KEY — ROTATION DAY 10: ROLLING MOTION

Rolling Without Slipping | For teacher use only

WARM-UP (PAGE 1)

(a) What determines whether a box slides or tips?

It depends on where you push and the friction coefficient.

If you push at the top edge, you create torque about the contact corner (pivot). If this torque exceeds the restoring torque from gravity (which acts at the center of mass), the box tips. If the push force exceeds the maximum static friction, the box slides instead. Which happens first depends on height, width, and friction.

(b) Why is a circle fundamentally different from a box?

A circle never has a stable equilibrium against tipping.

A box can resist tipping because the center of mass is above a flat base — gravity provides a restoring torque. A circle's center of mass is always directly above the contact point (the pivot). Therefore, gravity exerts zero restoring torque. Any small push causes the circle to tip — which we call rolling. Circles are in constant "unstable equilibrium," perpetually tipping over a moving pivot.

THE TIPPING THRESHOLD (PAGE 2)

(a) Condition for the box to tip

For the box to tip to the right, the applied torque must exceed the gravitational restoring torque, both measured about the pivot (front corner):

$$\tau_{\text{push}} > \tau_{\text{gravity}}$$

$$F \cdot h > mg \cdot \frac{w}{2}$$

$F > \frac{mgw}{2h}$

Here, h is the height where you push, and w is the box width.

(b) Restoring torque for the circle

For a circle, gravity acts straight down from the center of mass, passing through the contact point (the pivot). The lever arm is zero:

$$\tau_{\text{gravity}} = mg \times 0 = 0$$

The gravitational torque on a circle is always zero.

Therefore, any applied force $F > 0$ creates a positive torque about the pivot, causing angular acceleration. The circle will always rotate forward. This is why circles roll — they have no restoring torque to prevent tipping.

THE PARADOX: DOES THE RAMP DO WORK? (PAGE 3)

Key Insight: Zero Work at the Contact Point

The contact point is momentarily at rest ($v = 0$). Both the normal force and static friction act at this point. Since $W = F \cdot \Delta x$ and the contact point doesn't move, these forces do zero work.

Therefore, mechanical energy is conserved. Gravitational PE is redirected into translational and rotational KE by the ramp, but the ramp itself does no mechanical work (though static friction does apply a torque to spin the object).

THE MASTER SHAPE TABLE (PAGE 3)

Problem: Complete the table for each shape rolling down a ramp of height h.

Formula: $v = \sqrt{\frac{2gh}{1+c}}$ **where** $c = I/(MR^2)$

Energy fractions: $K_{\text{rot}}/K_{\text{total}} = c/(1+c)$ **and** $K_{\text{trans}}/K_{\text{total}} = 1/(1+c)$

Solid Sphere ($c = 2/5$**):**

$$v = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{2gh}{7/5}} = \sqrt{\frac{10gh}{7}}$$

For h = 1.0 m: $v = \sqrt{\frac{10(9.80)(1)}{7}} = \sqrt{14} = 3.74\ \text{m/s}$

$$K_{\text{rot}}/K_{\text{total}} = \frac{2/5}{1+2/5} = \frac{2/5}{7/5} = \frac{2}{7} = 28.6\%$$

$$K_{\text{trans}}/K_{\text{total}} = \frac{1}{1+2/5} = \frac{5}{7} = 71.4\%$$

Rank: 4th (slowest)

Solid Cylinder ($c = 1/2$**):**

$$v = \sqrt{\frac{2gh}{1+1/2}} = \sqrt{\frac{2gh}{3/2}} = \sqrt{\frac{4gh}{3}}$$

For h = 1.0 m: $v = \sqrt{\frac{4(9.80)}{3}} = \sqrt{13.07} = 3.61\ \text{m/s}$

$$K_{\text{rot}}/K_{\text{total}} = \frac{1/2}{3/2} = \frac{1}{3} = 33.3\%$$

$$K_{\text{trans}}/K_{\text{total}} = \frac{2}{3} = 66.7\%$$

Rank: 3rd

Thin Hoop ($c = 1$**):**

$$v = \sqrt{\frac{2gh}{1+1}} = \sqrt{\frac{2gh}{2}} = \sqrt{gh}$$

For h = 1.0 m: $v = \sqrt{9.80} = 3.13\ \text{m/s}$

$$K_{\text{rot}}/K_{\text{total}} = \frac{1}{2} = 50\%$$

$$K_{\text{trans}}/K_{\text{total}} = \frac{1}{2} = 50\%$$

Rank: 2nd

Sliding Block ($c = 0$**):**

$$v = \sqrt{\frac{2gh}{1+0}} = \sqrt{2gh}$$

For h = 1.0 m: $v = \sqrt{2(9.80)} = \sqrt{19.6} = 4.43\ \text{m/s}$

All energy is translational (100%)

Rank: 1st (fastest)

Table Summary

Shape	c value	Speed at h=1m	% Rotational KE	% Trans. KE	Rank
Solid sphere	2/5 = 0.40	3.74 m/s	28.6%	71.4%	4th
Solid cylinder	1/2 = 0.50	3.61 m/s	33.3%	66.7%	3rd
Thin hoop	1	3.13 m/s	50%	50%	2nd
Sliding block	0	4.43 m/s	0%	100%	1st

Pattern Recognition

(a) Which shape wastes the most and least energy on spinning?

Hoop wastes the most (50% on rotation); sphere wastes the least (28.6%).

The hoop has all its mass at the rim ($I = MR^2$), so it has the largest c. More rotation means more of the energy goes into spin instead of translation. The sphere has mass distributed closer to the center ($I = \frac{2}{5}MR^2$), so it "spins more efficiently."

(b) Solid disk always splits KE as ___% translational, ___% rotational

66.7% translational (or 2/3), 33.3% rotational (or 1/3).

This is a key fact to memorize. Every solid cylinder/disk rolling without slipping splits its kinetic energy exactly this way, independent of mass, radius, or speed.

(c) Would a hollow sphere (c = 2/3) finish faster or slower?

Slower than a solid sphere (c = 2/5), but faster than a hoop (c = 1).

A hollow sphere has $c = 2/3$, which is between 2/5 and 1. Using $v = \sqrt{\frac{2gh}{1+2/3}} = \sqrt{\frac{6gh}{5}}$, we get about 3.43 m/s for h = 1 m, which ranks between the solid sphere and hoop.

WHEN ENERGY WORKS — SCENARIO A: ROLLING WITHOUT SLIPPING (PAGE 4)

Friction type: Static friction (contact point doesn't slide)

Energy conserved? YES. Use $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ with $v = \omega R$.

SCENARIO B: SLIDING THEN ROLLING (PAGE 4)

Friction type: Kinetic friction (contact point is rubbing/sliding)

Energy conserved? NO. Kinetic friction does work ($W = f_k \times \text{distance slid}$) and converts mechanical energy to heat.

WHICH SCENARIO? (PAGE 4)

Situation	Friction Type	Energy Conserved?
Ball rolls down rough ramp from rest	**Static friction**	**YES**
Bowling ball thrown with no spin	**Kinetic friction**	**NO**

Ball on frictionless surface	**None**	**YES**

HOMEWORK

1. Energy Split Shortcut

Solid cylinder rolling at v_cm = 6.0 m/s, m = 2.0 kg

For a solid cylinder, $c = 1/2$:

$$K_{\text{total}} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(1+c)mv_{cm}^2 = \frac{1}{2}(1.5)(2.0)(6.0)^2 = 1.5 \times 36 = 54 \text{ J}$$

$$K_{\text{trans}} = \frac{1}{1+c}K_{\text{total}} = \frac{2}{3}(54) = 36 \text{ J}$$

$$K_{\text{rot}} = \frac{c}{1+c}K_{\text{total}} = \frac{1}{3}(54) = 18 \text{ J}$$

K_trans = 36 J | K_rot = 18 J

2. The Bowling Ball

Solid sphere: m = 6.0 kg, R = 0.11 m, v_0 = 8.0 m/s (no spin), μ_k = 0.10

(a) What force slows the center of mass?

Kinetic friction (acting opposite to motion direction). The contact point is sliding backward relative to the ball's center, so kinetic friction acts forward on the center (but wait — friction acts backward on the contact point, which opposes forward motion). Actually, kinetic friction acts **backward** (opposite the velocity), decelerating the center of mass.

(b) What force causes the ball to start spinning?

Kinetic friction (same force, different effect). At the contact point, kinetic friction acts backward (opposite motion). This creates a torque about the center that causes the ball to spin. It's the same friction force, but its effect on translation vs. rotation depends on where it acts.

(c) Challenge: When v_cm = ωR, find the final rolling speed.

$$f_k = \mu_k mg = 0.10(6.0)(9.80) = 5.88 \text{ N}$$

$$a_{cm} = -f_k/m = -5.88/6.0 = -0.98 \text{ m/s}^2$$

$$\tau = f_k R = 5.88(0.11) = 0.647 \text{ N·m}$$

$$I = \frac{2}{5}mR^2 = \frac{2}{5}(6.0)(0.11)^2 = 0.0290 \text{ kg·m}^2$$

$$\alpha = \tau/I = 0.647/0.0290 = 22.3 \text{ rad/s}^2$$

At time t: $v(t) = 8.0 - 0.98t$ and $\omega(t) = 22.3t$

Rolling constraint $v = \omega R$ is satisfied when:

$$8.0 - 0.98t = 22.3t(0.11) = 2.45t$$

$$8.0 = 3.43t \implies t = 2.33 \text{ s}$$

$$v_{\text{final}} = 8.0 - 0.98(2.33) = 8.0 - 2.28 = 5.72 \text{ m/s}$$

Ratio: $5.72/8.0 = 0.715 \approx 5/7$ ✓

Final rolling speed is $\frac{5}{7}v_0 = 5.7$ m/s. The lost energy $\Delta KE = \frac{1}{2}(6.0)(8.0)^2 - \frac{1}{2}(6.0)(5.7)^2 = 192 - 97.7 = 94.3$ J went to heat generated by sliding friction.

3. Will It Tip or Will It Slide? (Page 5)

Box: width = 0.40 m, height = 1.0 m, mass = 20 kg, μ_s = 0.35, pushing at top edge

(a) Minimum force to slide:

Sliding occurs when the applied force exceeds maximum static friction:

$$F_{\text{slide}} = \mu_s mg = 0.35(20)(9.80) = 68.6\text{ N}$$

(b) Minimum force to tip:

Tipping about the pivot (front corner) requires applied torque to exceed gravitational restoring torque:

$$F \times h = mg \times \frac{w}{2}$$

$$F_{\text{tip}} = \frac{mg \times w/2}{h} = \frac{20(9.80)(0.20)}{1.0} = 39.2\text{ N}$$

(c) Which happens first?

Tipping happens first (39.2 N < 68.6 N). When you push at the top, tipping is easier than sliding.

If you pushed at the midpoint (h = 0.50 m):

$$F'_{\text{tip}} = \frac{20(9.80)(0.20)}{0.50} = 78.4\text{ N}$$

Now $F'_{\text{tip}} = 78.4 > 68.6 = F_{\text{slide}}$, so sliding happens first. This is why pushing lower on an object is more likely to make it slide instead of tip.