ROTATION DAY 8: ROTATIONAL INERTIA BY INTEGRATION

WARMUP Day 7 Bridge — the Discrete Sum

Two 2.0 kg masses sit at the ends of a light 1.0 m rod (each 0.50 m from the center). Find the rotational inertia about the center axis using $I = \Sigma mr^2$.

Yesterday's table values were somebody's integral. $ML^2/12$, $\frac{1}{2}MR^2$, $\frac{2}{5}MR^2$ — every entry came from the same machine you used for the center of mass in Unit 4: chop the object into dm slices. The only change: each slice is now weighted by **r^2**, its squared distance from the AXIS, because distance from the axis is what resists spin.

THE INERTIA INTEGRAL

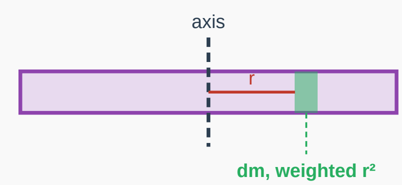
Weight every slice by r^2 and accumulate:

$$I = \int r^2\, dm$$

For a uniform rod ($\lambda = M/L$) spun about its CENTER, the slices run from $-L/2$ to $+L/2$:

$$I = (M/L) \int_{-L/2}^{L/2} x^2\, dx = (M/L)\cdot(L^3/12) = \mathbf{ML^2/12}$$

Units check: each strip carries (kg)(m²) = kg·m². This integral is where the table value comes from.



WE DO Exercise 1: Run the Rod Integral

A uniform rod has $M = 6.0$ kg and $L = 2.0$ m. **(a)** Write λ with units. **(b)** Run $I = (M/L)\int x^2 dx$ from $-L/2$ to $+L/2$, showing every step. **(c)** Confirm your number against $ML^2/12$.

SAME ROD, DIFFERENT AXIS: WHY THE END COSTS 4×

ROD ABOUT ITS END

Move the axis to the END of the rod and the slices run from 0 to L:

$$I = (M/L) \int_0^L x^2\,dx = (M/L)\cdot(L^3/3) = \mathbf{ML^2/3}$$

Four times the center value — with the axis at the end, the far half of the rod sits at large r, and the r^2 weighting makes distant mass count heavily. The farthest slice alone counts $(L)^2$ vs the center-spin's $(L/2)^2$.

I is not a property of the object alone. It is a property of *object* + *axis*. Never quote "the" rotational inertia of a rod without saying where it spins.

WE DO **Exercise 2: The Same Rod, Twice**

For the M = 6.0 kg, L = 2.0 m rod: **(a)** Compute I about the end with the integral. **(b)** Divide by your Exercise 1 answer and confirm the factor of 4. **(c)** In one sentence: where did the extra inertia come from, in r^2 language?

YOU DO **Exercise 3: The Easiest Integral You'll Ever Do**

A hoop of mass M and radius R spins about its central axis. Every dm sits at the SAME distance r = R. **(a)** Pull R^2 out of $I = \int r^2 dm$ and finish the integral in one line. **(b)** Why is the hoop the "worst case" — the largest I any shape of mass M and outer radius R can have?

THE PARALLEL-AXIS THEOREM: CONNECTING THE TWO RESULTS

$I = I_{CM} + MD^2$

Spin an object about any axis parallel to one through its center of mass, a distance d away:

$$I = I_{cm} + Md^2$$

The punchline for today's rod: end axis = center axis shifted by $d = L/2$, so

$$I_{end} = ML^2/12 + M(L/2)^2 = ML^2/12 + 3ML^2/12 = \mathbf{ML^2/3} \checkmark$$

Your two integrals from pages 1-2 were secretly one theorem apart. Note the theorem only jumps FROM the cm axis — never between two arbitrary axes.

WE DO **Exercise 4: Verify Numerically**

For the $M = 6.0$ kg, $L = 2.0$ m rod: plug into $I_{cm} + Md^2$ with $d = 1.0$ m and confirm it reproduces your Exercise 2 answer exactly.

CALC **Exercise 5: A Disk on Its Rim**

A uniform disk ($M = 4.0$ kg, $R = 0.50$ m) has $I_{cm} = ½MR^2$ about its center. It is pinned at a point on its RIM and swings about that pin. **(a)** Find I about the pin with the parallel-axis theorem. **(b)** Express the result as a multiple of MR^2.

PAPER PRACTICE

PRACTICE PROBLEMS

P1 Rod, Center Axis

A uniform rod has $M = 3.0$ kg and $L = 4.0$ m. Compute I about its center — run the integral, then check against $ML^2/12$.

P2 Rod, End Axis — Your Choice of Route

Same rod ($M = 3.0$ kg, $L = 4.0$ m), axis at the end. Compute I twice: once by integral, once by parallel-axis from P1. The answers must agree.

P3 Hoop vs Disk

A hoop and a uniform disk have the SAME mass M and radius R. **(a)** Write both I values about the central axis. **(b)** Same torque applied to each — which spins up faster, and by what factor? ($\alpha = \tau/I$.)

P4 The Loaded Rod Returns

A rod of length L has $\lambda(x) = cx$ (light at $x = 0$, dense at $x = L$), spun about the LIGHT end ($x = 0$). **(a)** Compute $I = \int_0^L x^2 \cdot \lambda \, dx$ and $M = \int_0^L \lambda \, dx$ symbolically; show $I = ML^2/2$ with the c canceling. **(b)** Compare to the uniform rod's $ML^2/3$ about its end — one sentence on why the loaded rod resists MORE, in strip language.