OSCILLATIONS DAY 2: SHM AS A DIFFERENTIAL EQUATION

WARMUP **Day 1 Bridge**

A 2.0 kg cart on a spring ($k = 8.0$ N/m) sits at $x = +0.50$ m from equilibrium. Use $F = -kx$ and Newton's second law to find its acceleration (sign matters).

Yesterday's equation is secretly a sentence about derivatives. $F = -kx$ with $F = ma$ says: *the second derivative of position is proportional to negative position* — $\mathbf{d^2x/dt^2 = -(k/m)x}$. An equation whose unknown is a FUNCTION is a differential equation, and this one is the most famous in physics. Today we solve it the way physicists actually do: guess wisely, then verify by differentiating.

VERIFY THE GUESS

Claim: $\mathbf{x(t) = A\cos(\omega t)}$ solves $d^2x/dt^2 = -\omega^2 x$. Check it — two derivatives:

$$x = A\cos(\omega t) \;\rightarrow\; v = dx/dt = -A\omega\sin(\omega t) \;\rightarrow\; a = dv/dt = -A\omega^2\cos(\omega t)$$

$$\mathbf{a = -\omega^2 \cdot [A\cos(\omega t)] = -\omega^2 x} \;✓$$

The cosine is verified by substitution, not assumed. Any A works (the equation is satisfied for every amplitude), and matching $-\omega^2 x$ to $-(k/m)x$ forces $\omega^2 = k/m$.

WE DO **Exercise 1: Run the Verification**

Take $x(t) = 0.20\cos(5t)$ (meters, seconds). **(a)** Differentiate twice to get $v(t)$ and $a(t)$. **(b)** Show $a = -25x$. **(c)** Read off ω and state what spring-mass combinations (k/m ratio) this motion could belong to.

WHAT THE SOLUTION HANDS YOU FOR FREE

Ω, T, V_MAX, A_MAX — WHERE THEY COME FROM

Matching the verified solution to the spring equation:

$$\boldsymbol{\omega = \sqrt{k/m}} \qquad T = 2\pi/\omega = 2\pi\sqrt{m/k}$$

And the derivatives you just took carry the speed and acceleration limits inside them: sin and cos never exceed 1, so

$$\boldsymbol{v_{max} = A\omega} \text{ (at the midpoint)} \qquad \boldsymbol{a_{max} = A\omega^2} \text{ (at the endpoints)}$$

Units audit: A(m)·ω(1/s) = m/s ✓ and A·ω² = m/s² ✓. Note what's missing: the period does NOT depend on A.

WE DO **Exercise 2: From Spring to Numbers**

A 2.0 kg cart oscillates on a $k = 50$ N/m spring with amplitude $A = 0.40$ m. **(a)** Find ω and T. **(b)** Find v_{max} and a_{max}. **(c)** WHERE on the track does each maximum occur?

YOU DO **Exercise 3: Read It All Off One Function**

A block's motion is $x(t) = 0.30 \cos(4t)$ (meters, seconds). Find **(a)** the amplitude, **(b)** ω and T, **(c)** v_{max}, and **(d)** a_{max}.

X, V, A: ONE CYCLE, THREE CURVES

CALCULUS CORNER — THE PHASE STORY

The three curves are the SAME cosine family, each a derivative apart:

$$x = A\cos(\omega t) \qquad v = -A\omega\sin(\omega t) \qquad a = -A\omega^2\cos(\omega t)$$

Read the geometry: **a is exactly opposite x** (the restoring force in graph form), and **v runs a quarter-cycle ahead**. So at the endpoints ($x = \pm A$): $v = 0$ and $|a|$ is maximal; at the midpoint ($x = 0$): $|v|$ is maximal and $a = 0$. Tomorrow's graph-reading day is these three lines.

WE DO

Exercise 4: Evaluate at an Instant

For $x(t) = 0.50\cos(2t)$ (meters, seconds), evaluate x, v, and a at $t = \pi/4$ s. ($\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$.) Then say where the block is and what it is doing at that moment.

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Exercise 5: Backwards From the Maxima

A sensor reports an oscillator's $v_{max} = 3.0$ m/s and $a_{max} = 12$ m/s² — nothing else. **(a)** Find ω from the ratio a_{max}/v_{max}. **(b)** Find the amplitude A. **(c)** Which pair of formulas from page 2 did you just invert?

PAPER PRACTICE

PRACTICE PROBLEMS

P1 The Sine Works Too

Verify that $x(t) = A \sin(\omega t)$ ALSO solves $d^2x/dt^2 = -\omega^2 x$ by differentiating twice. Then answer: physically, what is different about an oscillator described by sine instead of cosine? (Hint: where is it at $t = 0$?)

P2 Stiff Spring, Light Cart

A 0.50 kg cart on a $k = 200$ N/m spring oscillates with amplitude 0.10 m. Find ω, v_{max}, and a_{max}.

P3 A Turning Point, Caught in the Act

For $x(t) = 0.40 \cos(3t)$ (meters, seconds), evaluate x and v at $t = \pi/3$ s. ($\cos(\pi) = -1$, $\sin(\pi) = 0$.) What special moment of the cycle did you land on?

P4 Tomorrow's Pendulum, Today's Equation

For small angles, a pendulum's restoring acceleration is $a = -(g/L)x$ — the SAME differential equation with g/L in place of k/m. **(a)** Write ω for the pendulum by analogy (no new derivation needed). **(b)** Compute T for $L = 2.45$ m with $g = 9.8$ m/s². **(c)** One sentence: why does the pendulum's T contain no mass while the spring's T does?