MOMENTUM DAY 6: CENTER OF MASS BY INTEGRATION

WARMUP **Unit 2 Bridge — the Weighted Average**

A 2.0 kg mass sits at x = 0 and a 6.0 kg mass sits at x = 4.0 m. Find the center of mass of the pair using $x_{cm} = (m_1x_1 + m_2x_2)/(m_1 + m_2)$.

From dots to solid objects. The weighted average above works for POINT masses. But a baseball bat, a rod, a bridge — mass is smeared continuously. The fix is the same move as every unit this year: chop the object into strips, weight each strip by its position, and let the sum become an integral.

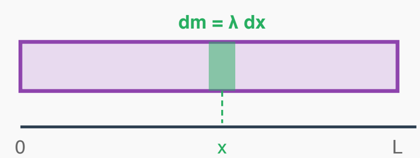
THE CENTER-OF-MASS INTEGRAL

Chop a rod into slices of mass dm at position x. Each slice contributes x·dm to the weighted sum:

$$x_{cm} = (1/M) \int x \, dm$$

For a rod along the x-axis, convert dm to dx with the **linear density** λ (kg/m): dm = λ dx. A UNIFORM rod has λ = M/L, and the integral gives what symmetry promised:

$$x_{cm} = (1/M)\int_0^L x\,(M/L)\,dx = (1/L)\cdot(L^2/2) = \mathbf{L/2}$$



WE DO **Exercise 1: The Integral Agrees With Your Eyes**

A uniform rod has length L = 4.0 m and mass M = 6.0 kg. **(a)** What is λ, with units? **(b)** Run the integral $x_{cm} = (1/M)\int_0^L x\lambda\, dx$, showing every step. **(c)** Confirm the answer matches the symmetry argument.

NON-UNIFORM RODS: WHERE SYMMETRY FAILS, THE INTEGRAL DOESN'T

A ROD THAT GETS HEAVIER TOWARD ONE END

Suppose $\lambda(x) = cx$ — the rod is light at $x = 0$ and dense at $x = L$. Two integrals, same strips:

$$M = \int_0^L cx\,dx = \mathbf{cL^2/2} \qquad \int_0^L x\cdot(cx)\,dx = cL^3/3$$

$$x_{cm} = (cL^3/3)/(cL^2/2) = \mathbf{2L/3}$$

The c cancels: EVERY linearly-loaded rod balances at two-thirds of its length, no matter how steep the loading. Sanity check: 2L/3 > L/2 — the CM shifted toward the heavy end, as it must.

Don't average the endpoints of λ. The CM is a mass-weighted average of POSITION — you must integrate x dm, then divide by the real total mass M (its own integral). Two integrals, always.

WE DO **Exercise 2: Run Both Integrals**

A rod of length $L = 3.0$ m has $\lambda(x) = 4x$ (kg/m when x is in meters). **(a)** Find the rod's total mass M. **(b)** Find x_{cm} by integrating x dm. **(c)** Check your answer against the 2L/3 shortcut.

YOU DO **Exercise 3: Your Turn, Full Pipeline**

A rod of length $L = 6.0$ m has $\lambda(x) = 2x$ (kg/m when x is in meters). **(a)** Find M. **(b)** Find x_{cm} with the integral. **(c)** One sentence: why is x_{cm} the same as Exercise 2's fraction of L even though the rods differ?

WHY THE CM RUNS THE MOMENTUM SHOW

THE CM DOESN'T CARE ABOUT INTERNAL FORCES

Differentiate the CM definition and multiply by M, and the system's total momentum appears:

$$p_{total} = M\, v_{cm}$$

Internal forces come in Newton's-third-law pairs, so they cancel in the total: only EXTERNAL forces change v_{cm}. An explosion, a collision, two skaters shoving each other — the pieces scatter, but the center of mass sails on exactly as before. That is what "conservation of momentum" looks like in space: **the CM's velocity is untouchable from the inside.**

WE DO Exercise 4: The CM Sails Through an Explosion

A 4.0 kg projectile moving at 6.0 m/s explodes into a 1.0 kg piece and a 3.0 kg piece. The 1.0 kg piece flies BACKWARD at 6.0 m/s. **(a)** Find the 3.0 kg piece's velocity from momentum conservation. **(b)** Compute v_{cm} of the two pieces after the blast and confirm it is still 6.0 m/s.

CALC Exercise 5: The CM That Refused to Move

Two skaters — 50 kg and 75 kg — stand at rest on frictionless ice and push off each other. The system's CM starts at rest, so it STAYS at rest. **(a)** After the 50 kg skater has glided 3.0 m one way, how far has the 75 kg skater glided the other way? **(b)** One sentence: what single fact about v_{cm} did you use?

PAPER PRACTICE

PRACTICE PROBLEMS

P1 Uniform, by Integral

A uniform rod has length $L = 8.0$ m and mass $M = 4.0$ kg. Compute x_{cm} with the integral $(1/M)\int_0^L x\lambda\, dx$ — show the λ you used and every step, even though you can guess the answer.

P2 Loaded Rod

A rod of length $L = 2.0$ m has $\lambda(x) = 6x$ (kg/m when x is in meters). **(a)** Find its total mass. **(b)** Find x_{cm}.

P3 CM Velocity, Before and After

A 2.0 kg cart moving at 5.0 m/s approaches a stationary 3.0 kg cart. **(a)** Find v_{cm} before they touch. **(b)** They collide and stick. Find the combined velocity — and say why you already knew it before computing.

P4 Prove the Shortcut

For a rod of length L with $\lambda(x) = cx$, prove symbolically that $x_{cm} = 2L/3$: set up both integrals, keep c as a letter, and show it cancels. Then answer: for $\lambda(x) = cx^2$ (even more end-loaded), should x_{cm} land closer to L or closer to 2L/3 — and why, in strip language?