KINEMATICS DAY 3: DERIVATIVES OF MOTION

WARMUP The Shrinking-Δt Ladder

A cart's position is **$x(t) = t^2$** (meters, seconds). Compute the average velocity — with units at every step — over: **(a)** t = 3 s to 4 s, **(b)** t = 3 s to 3.1 s, **(c)** t = 3 s to 3.01 s. **(d)** The answers are heading toward one number. What is it?

THE DERIVATIVE: INSTANTANEOUS RATE

Yesterday every velocity was an *average* over some Δt. The warmup shows what happens as Δt shrinks: the average rate settles toward a single value — the **instantaneous velocity**. That limit is the **derivative**:

$$v = dx/dt \quad \text{(slope of the tangent line on the x-t graph)} \qquad a = dv/dt$$

Power rule (all you need for polynomials): if $x(t) = c \cdot t^n$, then $dx/dt = c \cdot n \cdot t^{n-1}$. Term by term, constants vanish. (Your calc class may also write $x'(t)$ — same thing.)

Units don't change with the limit: dx/dt is still meters over seconds — m/s. Derivatives inherit the units of the ratio.

WE DO Exercise 1: The Pattern, By Example

In words: **multiply by the power, then lower the power by one.** The left pairs are worked — fill in the right pairs together. Units check: if x is in meters and t in seconds, every derivative entry must come out in m/s.

Given x(t)	v(t) = dx/dt	Given x(t)	v(t) = dx/dt
$4t$	4	$5t$	
t^2	$2t$	$4t^2$	
$3t^2$	$6t$	$2t^3$	
t^3	$3t^2$	7	

WE DO Exercise 2: First Derivatives of Motion

A cart's position is **$x(t) = 3t^2 + 2t$** (x in m, t in s). **(a)** Find $v(t) = dx/dt$, then evaluate v at t = 2.0 s. **(b)** Find $a(t) = dv/dt$. **(c)** Dimensional audit: every term of x(t) must come out in meters — so what units must the coefficients 3 and 2 carry? Check that your v(t) terms then come out in m/s.

dx/dt is not "d times x over d times t." It's one symbol: the limit of Δx/Δt as Δt → 0. If your calc class hasn't formally defined it yet, the warmup IS the definition — physics just got there first.

TURNING POINTS AND WHAT THE SIGNS MEAN

READING V AND A TOGETHER

Momentarily at rest: $v(t) = 0$. Solve it — those are the turning points (if v changes sign there).

$a \neq 0$ while $v = 0$ is normal: at the turnaround, velocity is zero but it's *changing* — that's exactly what a measures.

Speeding up: v and a have the *same* sign.

Slowing down: v and a have *opposite* signs.

"Negative acceleration" does NOT mean slowing down — it means a points in the negative direction.

WE DO **Exercise 3: Full x → v → a Workup**

A particle's position is $\mathbf{x(t) = t^3 - 6t^2 + 9t}$ (m, s), for $t \geq 0$. **(a)** Find v(t) and a(t). **(b)** Find every time the particle is momentarily at rest. **(c)** Find the particle's position at each of those times. **(d)** Find a at each of those times — is the particle's velocity about to become positive or negative at each one?

YOU DO **Exercise 4: Speeding Up or Slowing Down?**

For the same particle, $x(t) = t^3 - 6t^2 + 9t$: **(a)** at $t = 2.5$ s, compute v and a (with units) and state whether the particle is speeding up or slowing down. **(b)** Do the same at $t = 4.0$ s. **(c)** In one sentence: what rule did you use?

CALCULUS CORNER — UNITS OF EVERY COEFFICIENT

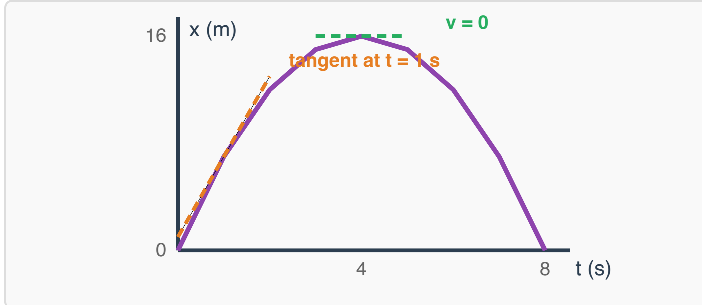
If $\mathbf{x(t) = At^3 + Bt}$ gives meters when t is in seconds, then A must carry **m/s³** and B must carry **m/s** — every term of a physics equation must land in the same units. Differentiate: $v(t) = 3At^2 + B \rightarrow (m/s^3)(s^2) = m/s$ and B = m/s. **The derivative's units take care of themselves — if the original equation was dimensionally honest.** Use this as a self-check on every derivative you take tonight.

CURVED GRAPHS: THE TANGENT LINE IS THE VELOCITY

READING A CURVED X-T GRAPH

On a curved x-t graph the slope is different at every instant. The velocity at time t is the slope of the **tangent line** there.

- Curve peaks (tangent flat) → v = 0, turning around
- Tangent steepening → |v| growing
- Concave down while rising → moving forward, slowing



WE DO **Exercise 5: The Graph and the Algebra Agree**

The graph above shows **$x(t) = 8t - t^2$** (m, s). **(a)** Find v(t) and use it to compute the slope of the tangent at t = 1 s — compare with the dashed orange line. **(b)** At what time is the tangent flat, and where is the particle then? **(c)** Find a(t). Does its sign match the curve's shape (concave down)?

CALC **Exercise 6: Slopes of Slopes**

Still $x(t) = 8t - t^2$: **(a)** make a quick table of v at t = 0, 2, 4, 6, 8 s. **(b)** Plot those five points as a v-t graph in the box — what shape do you get? **(c)** The slope of YOUR v-t graph is a(t): read it off, with units, and check it against dv/dt.

PAPER PRACTICE

PRACTICE PROBLEMS

P1 Straight to the Derivatives

A cart's position is $\mathbf{x(t) = 4t^2 - 3t + 2}$ (m, s). Find **(a)** v(t) and v at t = 1.5 s, and **(b)** a(t). **(c)** State the units of each coefficient (4, –3, 2) in x(t).

P2 Two Turning Points

A particle moves with $\mathbf{x(t) = 2t^3 - 9t^2 + 12t}$ (m, s), $t \geq 0$. **(a)** Find both times the particle is momentarily at rest. **(b)** Find the particle's position at each of those times. **(c)** Between those two times, which direction is the particle moving? Justify with a sign check of v.

P3 At Rest, Then Where?

For the cart of P1, $x(t) = 4t^2 - 3t + 2$: **(a)** find the one time its velocity is zero; **(b)** find the cart's position at that instant; **(c)** is the cart speeding up or slowing down just after? Justify with the signs of v and a.

P4 Dimensional Detective

A car's position is $\mathbf{x(t) = \frac{1}{2}bt^2}$ where b = 4.0 m/s². **(a)** Show the right side comes out in meters. **(b)** Find v(t) and a(t). **(c)** a(t) came out constant — what does that tell you about how the car's velocity changes? (Days 4 and 5 are devoted to exactly this special case — and b will earn a name.)

