ENERGY DAY 3: THE WORK INTEGRAL

WARMUP Day 2 Bridge

A force on a crate ramps in a straight line from 0 up to 24 N as the crate moves from x = 0 to x = 3.0 m. Find the work done by this force — area first, units of one strip attached.

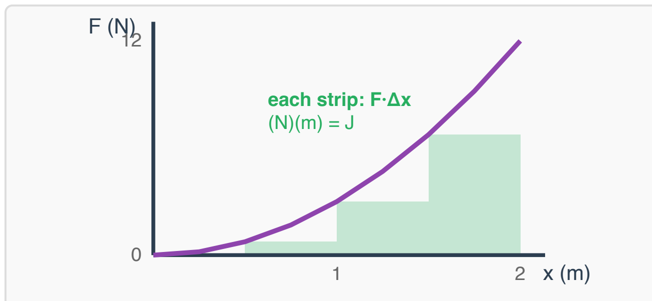
You already made this move once. In Unit 1 you shrank v-t strips until the sum became $\int v\,dt$. Today it is the exact same limit with new axes: force on the vertical, position on the horizontal. Yesterday's "break the area into shapes" becomes today's integral — the tool that handles curves no rectangle-and-triangle kit can touch.

FROM STRIPS TO THE WORK INTEGRAL

Take $F(x) = 3x^2$ (N, x in meters). Chop x = 0 → 2 m into strips of width Δx. Each strip adds **$F\cdot\Delta x$** of work — units (N)(m) = J. Add them up, then shrink Δx:

Strip width	Running total (left edges)
Δx = 1 m	(0 + 3)(1) = 3 J
Δx = 0.5 m	(0 + 0.75 + 3 + 6.75)(0.5) = 5.25 J
$\Delta x \to 0$	**8 J — the exact area**

That limit IS the work integral: $\mathbf{W = \int_0^2 F(x)\,dx = x^3 \big|_0^2 = 8\ J}$.



WE DO Exercise 1: Accumulate by Hand

For $F(x) = 3x^2$ above: **(a)** verify the Δx = 0.5 m running total (5.25 J) by adding the four strips with units. **(b)** Compute the exact work with the antiderivative and confirm 8 J. **(c)** Why does the strip total come out LOW here no matter how skinny the strips are — what about F(x) makes left edges undershoot?

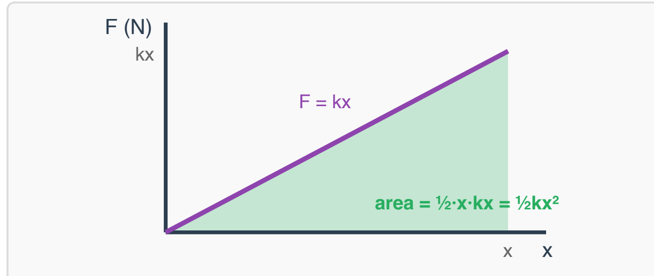
THE SPRING PAYOFF: DERIVING ELASTIC ENERGY

ACCUMULATING HOOKE'S LAW

In Unit 2 you measured that a spring pushes back with **$F = kx$**. How much work does it take to stretch one from 0 to a stretch x? Accumulate the force over the stretch:

$$W = \int_0^x kx' \, dx' = \tfrac{1}{2}kx^2$$

No memorizing — the formula is the **triangle's area** under $F = kx$: ½ · base (x) · height (kx) = $\tfrac{1}{2}kx^2$. The ½ is the same ½ that lives in $\tfrac{1}{2}at^2$: both are triangles left behind by a quantity that grows from zero.



Common trap: $W = Fd$ with $F = kx$ plugged in gives kx^2 — twice too big. That formula silently assumes the force was kx the WHOLE way, but the force starts at zero and grows. The accumulation keeps that honest.

WE DO **Exercise 2: The Second Stretch Costs More**

A spring has k = 400 N/m. **(a)** Find the work to stretch it from 0 to 0.20 m. **(b)** Find the work to stretch it from 0.20 m to 0.40 m. **(c)** Part (b) is three times part (a) — explain why in strip language (what is different about the strips on the second interval?).

YOU DO **Exercise 3: Coefficient Units First**

A force follows $F(x) = 5x$ (newtons when x is in meters). **(a)** What units does the coefficient 5 carry? **(b)** Find the work done by this force from x = 0 to x = 2.0 m by integrating. **(c)** Confirm your answer as a triangle area.

BEYOND SPRINGS: ANY F(X) YOU MEET

CALCULUS CORNER — THE ACCUMULATION TABLE GROWS

Graph	One strip is…	Units of one strip	So the area accumulates…
v-t	$v \cdot \Delta t$	(m/s)(s) = m	**displacement** (Δx)
a-t	$a \cdot \Delta t$	(m/s^2)(s) = m/s	**velocity change** (Δv)
F-x	$\mathbf{F \cdot \Delta x}$	**(N)(m) = J**	**work** (W)

Same habit as Unit 1: before you compute ANY area, write the units of one strip first. If the strip units aren't joules, whatever you are accumulating is not work.

WE DO **Exercise 4: Piecewise Work**

A force on a sled holds constant at 8.0 N for the first 2.0 m, then ramps in a straight line down to 0 over the next 2.0 m. **(a)** Accumulate each piece with units. **(b)** Total work over all 4.0 m.

YOU DO **Exercise 5: No Shape Kit for This One**

A magnetic launcher pushes a cart with $F(x) = 6x^2$ (newtons when x is in meters). **(a)** Why can't the rectangle-and-triangle method from Day 2 compute this area exactly? **(b)** Find the work done from x = 0 to x = 2.0 m by integrating.

CALC **Exercise 6: When the Area Goes Negative**

A force follows $F(x) = 10 - 4x$ (newtons when x is in meters) as an object moves from x = 0 to x = 4.0 m. **(a)** At what position does the force cross zero? **(b)** Find the total work by integrating over the whole interval. **(c)** The region past the zero crossing counts as negative area — what is the force doing to the object there, physically?

PAPER PRACTICE

PRACTICE PROBLEMS

P1 Straight Work Integral

A force follows $F(x) = 9x^2$ (newtons when x is in meters). Find the work it does from $x = 0$ to $x = 2.0$ m. Write the units of one strip before you integrate.

P2 Spring, Twice

A spring has $k = 250$ N/m. **(a)** Find the work to stretch it from 0 to 0.30 m. **(b)** Find the work to stretch it from 0.30 m to 0.60 m. **(c)** State the ratio of (b) to (a) — no new computation needed if you see the pattern from Exercise 2.

P3 Piecewise From a Description

A tow rope pulls with a constant 12 N for the first 3.0 m, then the tension ramps in a straight line down to 0 over the next 3.0 m. Sketch the F-x graph, then find the total work done by the rope over all 6.0 m.

P4 Two Terms, One Accumulation

A bungee cord pulls back with $F(x) = 20x + 6x^2$ (newtons when x is in meters). **(a)** What units do the coefficients 20 and 6 carry? **(b)** Find the work done against the cord stretching it from $x = 0$ to $x = 2.0$ m. **(c)** For a very large stretch, which term does most of the accumulating — and why does that make physical sense for a stiffening cord?